

ALGEBRAIC COMPACTNESS OF $\prod M_\alpha / \bigoplus M_\alpha$

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Abstract: In this note, we are working within the category $R\mathbf{Mod}$ of (unitary, left) R -modules, where R is a **countable** ring. It is well known (see e.g. Kiełpiński & Simson [5], Theorem 2.2) that the latter condition implies that the (left) pure global dimension of R is at most 1. Given an infinite index set A , and a family $M_\alpha \in R\mathbf{Mod}$, $\alpha \in A$ we are concerned with the conditions as to when the R -module

$$\prod / \prod = \prod_{\alpha \in A} M_\alpha / \bigoplus_{\alpha \in A} M_\alpha$$

is or is not algebraically compact. There are a number of special results regarding this question and this note is meant to be an addition to and a generalization of the set of these results. Whether the module in the title is algebraically compact or not depends on the numbers of algebraically compact and non-compact modules among the components M_α .

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Given an (infinite) cardinal κ , an R -module M is κ -compact, if, every system of $\leq \kappa$ linear equations over M (with unknowns x_j and almost all $r_{ij} = 0$):

$$\sum_{j \in J} r_{ij} x_j = m_i \in M, \quad i \in I, \quad r_{ij} \in R, \quad |I|, |J| \leq \kappa \quad (1)$$

has a solution in M whenever all its finite subsystems have solutions (in M). A module is (*algebraically*) *compact* if it is κ -compact, for every cardinal κ . It is well-known that if $M \in R\mathbf{Mod}$ is κ -compact, for some $\kappa \geq |R|$, then M is algebraically compact. Algebraic compactness of M is equivalent to pure injectivity and this in turn is equivalent to $\text{Pext}_R^1(X, M) = 0$, for every $X \in R\mathbf{Mod}$.

Recall that \prod / \coprod is a special case of a more general construction of the reduced product $\prod M_\alpha / \mathcal{F}$, where \mathcal{F} is the cofinite filter on A . Given a subset $B \subseteq A$, then $\mathcal{F} \cap B$ and $\mathcal{F} \cap (A \setminus B)$ are cofinite filters on B and on $A \setminus B$ respectively, if \mathcal{F} is the cofinite filter on A . One can now easily prove the following isomorphism (alternatively use Theorem 1.10 in [2]):

$$\prod_{\alpha \in A} M_\alpha / \bigoplus_{\alpha \in A} M_\alpha \cong \prod_{\alpha \in B} M_\alpha / \bigoplus_{\alpha \in B} M_\alpha \times \prod_{\alpha \in A \setminus B} M_\alpha / \bigoplus_{\alpha \in A \setminus B} M_\alpha. \quad (2)$$

The proof of the following result is straightforward, since it uses a powerful classical result of Mycielski.

Proposition 1. *For every countable index set B ,*

$$\prod / \coprod = \prod_{\alpha \in B} M_\alpha / \bigoplus_{\alpha \in B} M_\alpha$$

is an algebraically compact R -module.

Proof. Since B is countable, there is a countable family of cofinite subsets of B with empty intersection. By a classical result of Mycielski [6, Theorem 1), \prod / \coprod is \aleph_0 -compact. This is equivalent to its algebraic compactness, since the rings we consider here are countable. \square

Note that this result need not hold true, if R is uncountable. For instance, if K is a field and $R = K[[X, Y]]$ is the two-variable power series algebra, then $R^{\mathbb{N}}/R^{(\mathbb{N})}$ is not algebraically compact (see [4], Theorem 8.42).

Lemma 2. Assume that pure global dimension of R is ≤ 1 . If

E: $0 \longrightarrow A \xrightarrow{*} B \longrightarrow C \longrightarrow 0$ is a pure exact sequence and B is pure injective, then C is likewise pure injective (algebraically compact).

Proof. Given an arbitrary $X \in R\mathbf{Mod}$, the segment of the $\text{Pext}_R^1(X, \mathbf{E})$ exact sequence we are interested in is as follows: $\dots \longrightarrow \text{Pext}_R^1(X, B) \longrightarrow \text{Pext}_R^1(X, C) \longrightarrow \text{Pext}_R^2(X, A) \longrightarrow \dots$. Since $\text{puregld } R \leq 1$ we have $\text{Pext}_R^2(X, A) = 0$. Since B is pure injective, we have $\text{Pext}_R^1(X, B) = 0$. These facts now force $\text{Pext}_R^1(X, C) = 0$, i.e. C is pure injective. \square

Proposition 3. Let $\text{puregld } R \leq 1$ and let A be an arbitrary (infinite) index set; if every $M_\alpha, \alpha \in A$ is algebraically compact, then $\prod_{\alpha \in A} M_\alpha / \bigoplus_{\alpha \in A} M_\alpha$ is algebraically compact.

Proof. It is well known that $\prod = \bigoplus_{\alpha \in A} M_\alpha$ is a pure submodule of $\prod = \prod_{\alpha \in A} M_\alpha$ and that \prod is algebraically compact iff all the components M_α are algebraically compact. Appeal to Lemma 2 completes the proof. \square

Theorem 4. Given any index set A , let $B \subseteq A$ be (at most) a countable set and $\forall \alpha \in B, M_\alpha$ is not algebraically compact, while $\forall \alpha \in A \setminus B, M_\alpha$ is algebraically compact. Then

$$\prod / \prod = \prod_{\alpha \in A} M_\alpha / \bigoplus_{\alpha \in A} M_\alpha$$

is algebraically compact.

Proof. By Proposition 1, the R -module $\prod_{\alpha \in B} M_\alpha / \bigoplus_{\alpha \in B} M_\alpha$ is algebraically compact. By Proposition 3, $\prod_{\alpha \in A \setminus B} M_\alpha / \bigoplus_{\alpha \in A \setminus B} M_\alpha$ is likewise algebraically compact. Now use isomorphism (2) to conclude that \prod / \prod is algebraically compact. \square

Our main concern is the converse of Theorem 4: If \prod / \prod is algebraically compact, can we conclude that at most countably many M_α 's are not algebraically compact?

Every linear system (1) has a short-hand representation $\mu \cdot \mathbf{x} = \mathbf{m}$, where $\mu = (r_{ij})_{i \in I, j \in J}$ is the corresponding row-finite matrix (call it the system matrix) and $\mathbf{x} = (x_j)_{j \in J}$, $\mathbf{m} = (m_i)_{i \in I}$ are the corresponding column vectors. The rows of matrix μ (which are the left hand sides of equations (1)) may be viewed as elements of the free R -module $\bigoplus_{j \in J} Rx_j$. The cardinality of these R -modules is $|R|2^{|J|}$. Thus the cardinality of the

set of different matrices μ representing (left-hand-sides) of (1) is at most $(|R|2^{|J|})^{|I|} = |R|^{|I|}2^{|J||I|}$. For purposes of algebraic compactness, it suffices to consider only $|I| = |J| = \max(|R|, \aleph_0)$, thus the latter cardinality is at most $\max(2^{|R|}, 2^{\aleph_0})$; for countable rings this bound is 2^{\aleph_0} . This is an important fact that we use in the proof of the next result.

Proposition 5. *Let $|A| > \max(2^{|R|}, 2^{\aleph_0})$ and $\forall \alpha \in A$, M_α is not algebraically compact. Then \prod / \oplus is not algebraically compact.*

Proof. For every M_α , $\alpha \in A$, there is a system of equations of type (1)

$$S_\alpha : \sum_{j \in J} r_{ij}^\alpha x_j^\alpha = m_i^\alpha \in M_\alpha, \quad i \in I, \quad r_{ij} \in R, \quad |I| = |J| = \max(|R|, \aleph_0) \quad (3)$$

with the corresponding row finite system matrices $\mu_\alpha = (r_{ij}^\alpha)_{i \in I, j \in J}$ and the property that every finite subsystem is solvable, without the whole system being solvable. By the observation on the number of different system matrices μ_α , the number of different left hand sides of systems S_α is $\max(2^{|R|}, 2^{\aleph_0})$. By the assumption on the cardinality of A , we conclude that there are $|A|$ many systems S_α with identical left hand sides. Without loss of generality we assume this is correct for all $\alpha \in A$, thus we consider systems (3) where the coefficients $r_{ij}^\alpha = r_{ij}$ do not vary by coordinates $\alpha \in A$. This coefficient uniformity enables a passage to the induced system in $\prod M_\alpha / \oplus M_\alpha$:

$$S : \sum_{j \in J} r_{ij} \overline{(x_j^\alpha)_{\alpha \in A}} = \overline{(m_i^\alpha)_{\alpha \in A}} \quad i \in I, \quad (4)$$

(bars denote the classes mod $\oplus_{\alpha \in A} M_\alpha$). Every finite subsystem of S is equivalent to the set of coordinate finite subsystems of S_α , for all but finitely many $\alpha \in A$. These have solutions, which will be the coordinates of the solutions of the original finite subsystem of S . But S has no global solution, for if $x_j = \overline{(s_j^\alpha)_{\alpha \in A}}, j \in J$ were global solutions of S , then $x_j^\alpha = s_j^\alpha, j \in J$ would provide global solutions of S_α , for almost all $\alpha \in A$. This contradiction then completes the proof that $\prod M_\alpha / \oplus M_\alpha$ is not algebraically compact. \square

As we have not succeeded in extending the latter result to all infinite $|A|$, we formulate the following

Conjecture. If $|A|$ is an uncountable index set of cardinality $\leq 2^{|R|}$ and all $M_\alpha \in R\mathbf{Mod}$, $\alpha \in A$, are not algebraically compact, then \prod / \prod is not algebraically compact. If this is true then, for countable rings R , \prod / \prod is algebraically compact if and only if all but countably many $M_\alpha \in R\mathbf{Mod}$, $\alpha \in A$ are algebraically compact.

Remarks. There are strong indications the conjecture is correct: Gerstner [3] proved that $\mathbb{Z}^A / \mathbb{Z}^{(A)}$ is algebraically compact, iff A is countable. A generalization follows for reduced powers of modules over countable rings: If $M \in R\mathbf{Mod}$ is not algebraically compact, then use Lemma 1.2 in [1] to conclude that if $M^A / M^{(A)}$ is algebraically compact then A must be countable. For Abelian groups, Rychkov [7] proved that \prod / \prod is algebraically compact if and only if A is countable. In fact, if \mathcal{S} denotes a set of system matrices with the property that for every $M \in R\mathbf{Mod}$ that is not algebraically compact, there is a $\mu \in \mathcal{S}$ that is a system matrix for a system proving algebraic non-compactness of M , let \mathfrak{n} denote minimal cardinality of all such systems. Close inspection of the proof of Proposition 1, *ibid.* seems to reveal that the RD-purity used there is not essential, namely that it may be replaced by purity (a condition always satisfied for Prüfer domains). In that case, if $|A| > \max(\mathfrak{n}, \aleph_0)$ and all M_α , $\alpha \in A$ are non-compact implies that \prod / \prod is non-compact.

References

- [1] B. Franzen, Algebraic compactness of filter quotients, *Proceedings Abelian Group Theory, Oberwolfach, 1981*, Lecture Notes in Math., Springer-Verlag **874**(1981), 228–241.
- [2] T. E. Frayne & A. C. Morel & D. S. Scott, Reduced direct products, *Fundamenta Mathematicae*, **51**(1962), 195–228.
- [3] O. Gerstner, Algebraische kompaktheit bei Faktorgruppen von Gruppen ganzzahliger Abbildungen, *Manuscripta math.*, **11**(1974), 103–109.
- [4] C. U. Jensen & H. Lenzing, *Model theoretic Algebra with particular emphasis on fields, rings, modules*, Gordon and Breach Science Publishers, New York (1989).
- [5] R. Kiełpiński & D. Simson, On pure homological dimension, *Bulletin de L'Acad. Polon. Sci., Sé. Math.*, **23**(1975), No.1, 1–6.
- [6] Jan Mycielski, Some compactifications of general algebras, *Colloquium Mathematicum*, **13**(1964), No.1, 1–9.

[7] S. V. Rychkov, On factor-group of the direct product of abelian groups modulo its direct sum, *Math. Notes*, **29**(1981), No.3-4, 252–257.

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